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On the existence of sports schedules with multiple venues

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Abstract

We give theoretical methods of creating sports schedules where there are multiple venues for the games, and the number of times each team uses each venue should be balanced. A construction for leagues having $2^p \geqslant 8$ teams was given by de Werra, Ekim and Raess. Here we show that feasible schedules exist when the league has an arbitrary even number of teams greater than or equal to 8.

Keywords: Sports scheduling; Kirkman's circle method

1. Introduction

Research on the construction of schedules for sports leagues covers a wide range of issues and techniques [1,2,5,7,10,11]. Perhaps the most popular type of problem concerns a league of teams having home stadiums, and the objective is to generate home-away patterns satisfying certain criteria. Integer and constraint programming and graph theory have been used to great effect in solving such problems.

The problem we consider here is not of this type; instead of having teams with fixed home stadiums, we have a set of $N = 2n$ teams and a set of n stadiums which have no association with the various teams. The objective is to create a schedule which satisfies the following five conditions:

(sched1) all team pairs play against each other at least once,
(sched2) on each day, n games involving all $2n$ teams are played,
(sched3) one game is played at each stadium on each day,
(sched4) each team plays 2 games at each of the n stadiums,
(sched5) each pair of teams plays no more than one game at each stadium.

The resulting schedule is different from the usual round robin in that each team must play $2n$ games, hence the scheduling period consists of $2n$ days, and for each team there is some partner with whom it meets twice. Urban and Russell [14,12] have applied various integer programming techniques and obtained solutions for $2n = 8$ and 16. On the theoretical side, de Werra et al. [3] have formulated this problem in terms of graphs, and constructed solutions for

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the values $2n = 2^p$ $(p \geqslant 3)$ by using inductive procedures (it is easy to see that the case $2n = 4$ has no solution). Their construction is based on the idea of using two disjoint subleagues to divide the scheduling period into days in which only games internal to the subleagues are held, and days in which interleague games are exclusively played. They have also proved that when $2n = 6$, no schedule satisfying conditions (sched1) to (sched5) exists.

We add here that a round robin schedule where each team uses each stadium at most twice, is called a *balanced tournament design*. Balanced tournament designs have been well studied, e.g. [6,8,9,13], and have been proved to exist for all $n \neq 2$ [6,13]. The problem of finding a balanced tournament design is related to, but not equivalent to the problem we treat in this paper, in the sense that the solution of one problem does not always give rise to a solution of the other. However, a special type of balanced tournament design, called a *Hamilton path tournament design* does in fact provide a solution to our problem, and their existence has been proved by Horton [8], when n is mutually prime with 30.

In this paper, we prove the following theorem:

Theorem. *Let n be any integer with* $n \geqslant 4$. *Then there always exists a schedule for* $2n$ *eams and n stadiums satisfying conditions* (*sched*1) *to* (*sched*5).

More precisely, we prove Theorems 1 and 2, and combine them with the results of [3].

Theorem 1. *Let n be any odd number greater than or equal to* 5. *Then there always exists a schedule for* $2n$ *teams and n stadiums satisfying conditions* (*sched*1) *to* (*sched*5).

Theorem 2. *Let n be any even number which is not a power of two*. *Then there always exists a schedule for* $2n$ *teams and n stadiums satisfying conditions* (*sched*1) *to* (*sched*5).

We break up the scheduling procedure into two parts: that of deciding for each team pair, when they play and where they play. Using terminology that is popular in sports scheduling, we will call the game days *slots* and the first part the *timetabling procedure*. A *timetable* is considered to be the assignment of teams pairs to slots, and will be represented by giving for each slot, the matches which are held on it. The second part involves assigning stadiums to each match after a timetable has been fixed. By regarding stadiums as *colors*, we will call this the *coloring procedure* in which we create a *color assignment*. A color assignment will be represented by specifying which color is given to each match. To simplify explanations in later sections, we will depart from normal conventions and consider both slot numbers and color indices as beginning from 0.

As in [3], our construction is based on dividing the $2n$ teams into disjoint subleagues. However, in our case, each subleague will consist of an odd number of teams, thus the number of subleagues is not always two; in fact, the odd numbers will be dominantly present throughout the whole paper. In the timetabling and coloring procedures, we make great use of Kirkman's circle method for creating round robin timetables, and also borrow some tools from elementary number theory. In Section 2, we define mathematical concepts, namely basic timetables and primitive roots that will be used later. In Section 3 we give our construction for odd n, and in Section 4, we treat even n which are not powers of two. Concluding remarks are given in Section 5.

2. Mathematical preliminaries

In this section, we introduce basic timetables and elementary facts of number theory.

2.1. Basic timetables

In the constructions which will be described, the polygon shown in Fig. 1 will play a central role. We consider odd n $(n \geqslant 3)$, and place n points at equal intervals along a circle. Beginning at the top, and proceeding clockwise, we assign *positions* $0, 1, 2, \ldots, n-1$ to the points, then produce the parallel lines which join positions 1 and $n-1$, 2 and $n-2$, and so on. We then give numbers $1, 2, \ldots, (n-1)/2$ to the lines so that line i $(1 \leqslant i \leqslant (n-1)/2)$ joins positions i and $n-i$, and say that line i has *depth* i. By natural association, we will say that the two positions i and $n-i$ also have depth i, and for convenience, we will sometimes refer to position 0 as the position having depth 0.

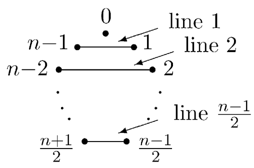

Fig. 1. Basic polygon.

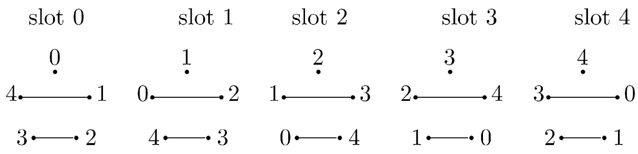

Fig. 2. Basic timetable for $n = 5$.

This polygon is well known as the basic tool used in Kirkman's circle method, which is a simple and ingenious way to create round robin timetables. For a league consisting of the n(odd) teams $\{0, 1, \ldots, n-1\}$, this method allows us to create a timetable consisting of n slots, in which of each, one team has no opponent(a bye), and the remaining $n-1$ teams are paired off into $(n-1)/2$ matches. First, we assign team i to position i on slot 0, and schedule matches between the two teams whose positions are joined by a line (the team assigned to position 0 has a bye). We then rotate the teams counterclockwise along the positions so that team s is assigned to position 0 on slot s. We will call the resulting timetable a *basic timetable* for n. The basic timetable for $n = 5$ is shown in Fig. 2.

We will call the team which is assigned to position i on slot s the *label* of i on slot s and denote it by $label(i, s)$. We observe the following facts concerning teams and labels in the basic timetable:

(lab1) for any team $t \in \{0, 1, \ldots, n-1\}$ and any position $i \in \{0, 1, \ldots, n-1\}$, there is a unique $s \in \{0, 1, \ldots, n-1\}$ with $label(i, s) = t$,

(lab2) each team $t \in \{0, 1, \ldots, n-1\}$ is assigned depth 0 exactly once and depth $d \in \{1, 2, \ldots, (n-1)/2\}$ exactly twice,

(lab3) $label(i, s) \equiv i + s \pmod n$,

(lab4) for any slot $s \in \{0, 1, \ldots, n-1\}$, the sum of the two labels of depth $d \in \{1, 2, \ldots, (n-1)/2\}$ is congruent to $2s \pmod n$,

(lab5) for any two distinct teams $t, t' \in \{0, 1, \ldots, n-1\}$, there is a unique $s \in \{0, 1, \ldots, n-1\}$ such that t and t' have the same depth on slot s.

Of course, the basic timetable can be constructed for any set containing an odd number of teams with some given ordering. For such a set T, the basic timetable for T will be understood to be the above basic timetable with suitable adaptations, likewise, $label(i, s)$ will be defined as the team of T assigned to position i on slot s.

2.2. *Prime numbers, primitive roots and indices*

We will briefly state some facts from elementary number theory which will be used later in the construction. For a more extensive background, see for example, Chapter 2 (especially pp. 43–49) of the book [4].

Let p be a prime number. For any $a \in \{1, 2, \ldots, p-1\}$, Fermat's theorem says that $a^{p-1} \equiv 1 \pmod p$. However, it may happen that $p-1$ is not the smallest exponent δ for which $a^{\delta} \equiv 1 \pmod p$. For example, when $p = 7$ and $a = 2$, we have

$$a^2 = 4, \quad a^3 = 8 \equiv 1 \pmod 7.$$

A *primitive root* of p is a number $r \in \{1, 2, \ldots, p-1\}$ such that $p-1$ is indeed the smallest exponent for which this occurs, or, equivalently, a number r such that the numbers $r, r^2, \ldots, r^{p-1}$ are all incongruent $\pmod p$. For example,

a	1	2	3	4	5	6
$\mathrm{Ind}_3\, a$	0	2	1	4	5	3

Fig. 3. Indices relative to the base $r = 3 (p = 7)$.

if $p = 7$ then for $r = 3$, we can easily verify that

$$r = 3, \quad r^2 \equiv 2, \quad r^3 \equiv 6, \quad r^4 \equiv 4, \quad r^5 \equiv 5, \quad r^6 \equiv 1 \pmod 7,$$

and $r = 3$ is a primitive root of $p = 7$. The following properties of primitive roots are well known.

Proposition 3 (*See [4, p. 46]*). *If p is prime, then p always has a primitive root. Moreover, if $p \geqslant 5$, then neither* 1 *nor $p - 1$ is a primitive root of p.*

Now let r be a primitive root of p. For any a which is not divisible by p, there is always a unique number $\alpha \in \{0, 1, \dots, p-2\}$ such that $a \equiv r^\alpha \pmod p$. (The exponent $p-1$ does not appear here as $r^{p-1} \equiv 1 = r^0$.) The exponent α is called the *index* of *a relative to the base r*, and denoted by $\mathrm{Ind}_r\, a$, or if there is no danger of confusion, simply as $\mathrm{Ind}\, a$. Fig. 3 shows the values of $\mathrm{Ind}_r\, a$ when $p = 7$ and $r = 3$. As may be expected, indices exhibit behavior similar to logarithms. The following facts hold for the indices relative to any primitive root of p [4, pp. 47–48].

(ind1) $a \equiv b \pmod p \iff \mathrm{Ind}\, a = \mathrm{Ind}\, b$,
(ind2) $\mathrm{Ind}(ab) \equiv \mathrm{Ind}\, a + \mathrm{Ind}\, b \pmod{p-1}$,
(ind3) $\mathrm{Ind}(a^m) \equiv m\, \mathrm{Ind}\, a \pmod{p-1}$,
(ind4) $p \neq 2 \Longrightarrow \mathrm{Ind}(p-1) = (p-1)/2$,
(ind5) $p \neq 2$ and $a + b = p \Longrightarrow \mathrm{Ind}\, a - \mathrm{Ind}\, b \equiv (p-1)/2 \pmod{p-1}$.

3. The case N=2n with n odd

In this section, we show that leagues with $N = 2n$ for odd $n \geqslant 5$ always allow schedules satisfying conditions (sched1) to (sched5). The construction will be divided into three cases: the first case will deal with the odd numbers which can be written as $n = 4q + 1$, the second with $n = 4q + 3$ and n prime, and the last with $n = 4q + 3$ and n composite.

3.1. *The case $n = 4q + 1$ $(q \geqslant 1)$*

We divide the $N = 2n$ teams into two subleagues whose teams we denote by $\{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$ and $\{0^*, 1^*, \dots, (n-1)^*\}$. We will call the first group of teams the *overbar teams* and the second group the *starred teams*. The set of colors is given by $\{C_0, C_1, \dots, C_{n-1}\}$. For each subleague, we create the basic timetable which we will call the overbar (starred) timetable. The polygons used will be referred to as the overbar (starred) polygons. The schedule will consist of two halves, each of which will be based on the overbar and starred timetables. Fig. 4 which shows the two halves of the schedule for $2n = 10 (n = 5)$ teams will serve as an example of the construction. The left-hand side represents the first half, and the right-hand side the second half. Slots proceed from top to bottom, and the boxed numbers shown on each line are the indices of the assigned colors.

Since $n = 4q + 1$, we notice that the number of lines $(n-1)/2 = 2q$ of the basic polygon is even. The timetable of the first half is constructed by taking the overbar and starred timetables, then exchanging the team labels on the left positions of lines $2, 4, \dots, 2q$ between the overbar and starred polygons on each slot. The timetable for the second half is constructed similarly, by flipping the labels on the left positions of lines $1, 3, \dots, 2q - 1$. In each half, the two teams with *label*$(0, s)$ on slot s, i.e., $\overline{s}$ and s^*, play against each other.

Claim 4. *In the two halves constructed above, all team pairs oppose each other exactly once, except for the pairs $(\overline{0}, 0^*), (\overline{1}, 1^*), \dots, (\overline{n-1}, (n-1)^*)$ which play against each other twice.*

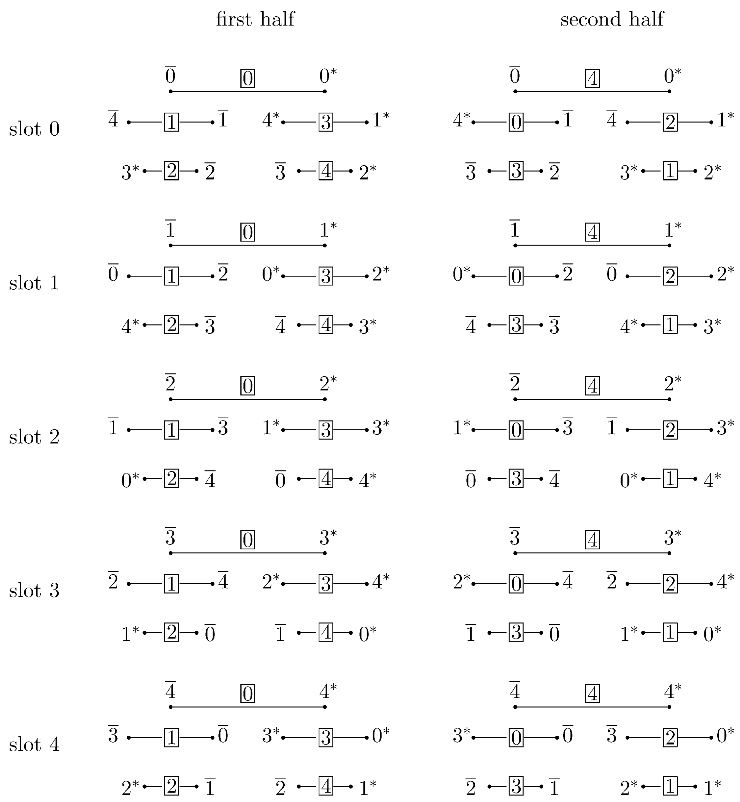

Fig. 4. Solution for 10 teams.

Proof. It is obvious that the pairs $(\overline{0}, 0^*)$, $(\overline{1}, 1^*)$, $\ldots$, $(\overline{n-1}, (n-1)^*)$ play twice. Now consider $t, t' \in \{0, 1, \ldots, n-1\}(t \neq t')$. By (lab5) the teams t and t' are assigned the same depth on some unique slot s in the basic timetable for n. If this depth is even, then the following two facts hold:

- The overbar teams $\overline{t}$ and $\overline{t'}$ oppose each other in the second half but not the first. The same can be said for the two starred teams t^* and $(t')^*$.
- The teams $\overline{t}$ and $(t')^*$ oppose each other in the first half but not the second. The same can be said for the pair t^* and $\overline{t'}$.

On the other hand, if the depth is odd, the above statements with the roles of first and second halves reversed, holds. This completes the proof of the claim. □

We now describe the color assignment. In both halves, each line in both the overbar and starred polygons will be given a fixed color which remains unchanged in all slots, and in each slot, each team is assigned the color given to the line it is incident to. For the first half, we give the two teams at depth 0 color C_0, then for $d = 1, 2, \ldots, 2q = (n-1)/2$, we assign to line d the color C_d in the overbar polygon, and the color C_{2q+d} in the starred polygon. Before describing the coloring for the second half, we make the following claim on the color assignment for the first half.

Claim 5. *The following three facts hold*: (i) *the colors* $C_0, C_2, C_4, \ldots, C_{4q}$ *are assigned to all teams in both* $\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ *and* $\{0^*, 1^*, \ldots, (n-1)^*\}$ *exactly once*, (ii) *all overbar teams are assigned* $C_1, C_3, \ldots, C_{2q-1}$ *twice, and no color from* $C_{2q+1}, C_{2q+3}, \ldots, C_{4q-1}$, *and* (iii) *all starred teams are assigned colors* $C_{2q+1}, C_{2q+3}, \ldots, C_{4q-1}$ *twice and no color from* $C_1, C_3, \ldots, C_{2q-1}$.

Proof. It is obvious that each team is assigned color C_0 exactly once. Moreover, (lab2) and the way the first half was constructed assure that each overbar (starred) team is assigned depth d exactly once in both overbar and starred polygons when d is even, and twice in the overbar (starred) own polygon when d is odd. The claims are then immediate from the way the colors are given to lines. □

The above claim can be interpreted as saying that colors given to lines which stand for interleague matches are assigned once each to all teams, while colors associated with lines which correspond to matches internal to subleagues are assigned twice each to the subleague teams. Considering the fact that in the second half, the lines with even depth correspond to the matches internal to subleagues, we see that it is sufficient to assign to these lines the colors $C_{2q+1}, C_{2q+3}, \ldots, C_{4q-1}$ in the overbar polygon, and the colors $C_1, C_3, \ldots, C_{2q-1}$ in the starred polygon. The remaining colors $C_0, C_2, C_4, \ldots, C_{4q}$ may be assigned arbitrarily as long as depth 0 receives a color different from C_0. A methodical way of achieving this is to tentatively assign color C_0 to the two teams with depth 0, then give colors for lines $d = 1, 2, \ldots, 2q$ by the following scheme:

$$\text{overbar polygon:} \begin{cases} C_{2q+d+1} & (d : \text{odd}), \\ C_{2q+d-1} & (d : \text{even}), \end{cases} \qquad \text{starred polygon:} \begin{cases} C_{d+1} & (d : \text{odd}), \\ C_{d-1} & (d : \text{even}), \end{cases}$$

and switch colors C_0 and $C_{4q} = C_{n-1}$ (this is the method used in Fig. 4). It is easy to see that the color assignment for the second half satisfies the following claim.

Claim 6. *The following three facts hold*: (i) *the colors* $C_0, C_2, C_4, \ldots, C_{4q}$ *are assigned to all teams in both* $\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ *and* $\{0^*, 1^*, \ldots, (n-1)^*\}$ *exactly once*, (ii) *all overbar teams are assigned* $C_{2q+1}, C_{2q+3}, \ldots, C_{4q-1}$ *twice, and no color from* $C_1, C_3, \ldots, C_{2q-1}$, *and* (iii) *all starred teams are assigned colors* $C_1, C_3, \ldots, C_{2q-1}$ *twice and no color from* $C_{2q+1}, C_{2q+3}, \ldots, C_{4q-1}$.

This completes the construction for $n = 4q + 1$.

Note that when $n = 4q + 3$, the number of lines is $2q + 1$ and hence an odd number. Thus the idea of 'pairing' the lines that we used above will not work.

Remark 7. The procedure we have described above is still valid if we substitute condition (sched5′), for (sched5). Indeed, the method by which we assigned colors to lines in the second half first constructs a coloring in which the repeated matches $(\overline{0}, 0^*), \ldots, (\overline{n-1}, (n-1)^*)$ are given color C_0 both times.

(sched5′) each pair of teams uses exactly one stadium for their game(s).

3.2. *The case* $n = 4q + 3$ $(q \geqslant 1)$ *and n is prime*

As we have mentioned in the introduction, this case follows from a result proved in [8]. However, as procedures described later will make use of results derived uniquely from our construction for this case, namely, Remark 14, we will describe it in full.

As in the case $n = 4q + 1$, we divide the $2n$ teams into the overbar and starred teams, $\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ and $\{0^*, 1^*, \ldots, (n-1)^*\}$, and create a schedule of two halves based on the overbar and starred timetables. However, in this case, the first half will consist exclusively of interleague matches, and the second half entirely of matches internal to subleagues (with the exception of matches between the teams assigned to depth 0). The schedule for $2n = 14 (n = 7)$ which is shown in Fig. 6 provides an example. As in Fig. 4, the left-hand side shows the first half and the right-hand side the second half, with slots proceeding from top to bottom, and the boxed numbers next to lines standing for the colors indices. In the following part, we identify n with 0, $n + 1$ with 1 and so on, in order to avoid lengthy explanations of trivial cases.

We first explain both the timetable and color assignment for the first half. The timetable is constructed by taking the overbar and starred timetables, then exchanging the labels on the left positions of all lines between the overbar and starred polygons in each slot. As before, the two teams $\overline{s}$ and s^* play against each other on slot s. The color assignment is again created by fixing colors to lines. We assign color C_{n-1} to the two teams with depth 0, and for line $d = 1, 2, \ldots, 2q + 1 = (n - 1)/2$, we associate color C_{d-1} in the overbar polygon and color C_{n-1-d} in the starred polygon. The following claim can be shown in a manner analogous to Claims 4 and 5.

Claim 8. *In the first half, the following facts hold*: (i) *every pair* $\overline{t} \in \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ *and* $(t')^* \in \{0^*, 1^*, \ldots, (n-1)^*\}$ *of teams plays against each other exactly once*, (ii) *each team is assigned each color exactly once*, *and* (iii) *in slot* 0, *the opponent of* $\overline{t}$ *is* $(n-t)^*$, *and the color* C_k *assigned to the match satisfies* $k \equiv t-1 \pmod n$ *for each* $t \in \{0, 1, \ldots, n-1\}$.

We now consider the second half. Again we take the basic overbar and starred timetables, but this time we change the team rotation on the starred polygons so that team $(n-s)^*$ is assigned depth 0 for $s = 0, 1, \ldots, n-1$ (see the right-hand side of Fig. 6). The following claim can be easily proved.

Claim 9. *The timetable for the second half satisfies the following facts*: (i) $\overline{s}$ *and* $(n-s)^*$ *is the unique interleague match occurring on slot* s, (ii) $\overline{t}$ *and* $\overline{t'}$ $(t \neq t')$ *are matched exactly once*, *likewise* t^* *and* $(t')^*$ *are matched exactly once*, *and* (iii) *if teams* $\overline{t}$ *and* $\overline{t'}$ *play against each other on slot* s, *then teams* t^* *and* $(t')^*$ *play against each other on slot* $n-s$ *and vice versa.*

We must now provide a method of assigning colors to teams so that each team receives each color exactly once in the second half. Clearly, the previous way of fixing colors to lines in the basic polygons will not work; instead for each slot s and each color C_k, we must choose one match on slot s, to which we assign color C_k. We begin by considering the problem of choosing matches to which we assign color C_0. On slot 0, we give this color to the match $(\overline{0}, 0)$. If we can then choose a set of $m = (n-1)/2$ line-slot pairs $\{(d_1, s_1), (d_2, s_2), \ldots, (d_m, s_m)\}$ in the basic timetable for n such that:

- each $t \in \{1, 2, \ldots, n-1\}$ appears exactly once in the set consisting of the labels of both ends of line d_i on slot s_i for $i = 1, 2, \ldots, m$,
- one line from each depth is chosen, i.e. $\{d_1, d_2, \ldots, d_m\} = \{1, 2, \ldots, m\}$,
- the set $\{s_1, s_2, \ldots, s_m\}$ contains exactly one of the pair s and $n-s$ for $s = 1, 2, \ldots, m$,

then, we can use this set to color matches in the following manner. (The set {(line 3, slot 1), (line 1, slot 2), (line 2, slot 4)} shown by thick lines in Fig. 5 satisfy the above three conditions for $n = 7$.)

Let $t_1 = label(d_1, s_1)$, $t_2 = label(n-d_1, s_1), \ldots, t_{2m-1} = label(d_m, s_m)$, $t_{2m} = label(n-d_m, s_m)$, i.e., t_{2i-1} and t_{2i} are the labels on the right and left positions of line d_i on slot s_i (in the basic timetable), for $i = 1, 2, \ldots, m$. Now color all pairs $(\overline{t_1}, \overline{t_2}), \ldots, (\overline{t_{2m-1}}, \overline{t_{2m}})$ in the overbar timetable and all pairs $(t_1^*, t_2^*), \ldots, (t_{2m-1}^*, t_{2m}^*)$ in the starred timetable with color C_0. By the first condition on the line-slot pairs, all teams $\overline{1}, \ldots, \overline{n-1}, 1^*, \ldots, (n-1)^*$ receive color C_0 exactly once, and the third condition and fact (iii) of Claim 9 guarantee that exactly one match in each slot is colored by C_0. Thus we have successfully assigned color C_0.

We now consider color C_1. We first color the pair $(\overline{1}, (n-1)^*)$, that is, the pair assigned to depth 0 on slot 1 by C_1. Then, for $i = 1, \ldots, m$, we color the two teams of depth d_i on slot $s_i + 1$ in the overbar timetable, and the two teams of depth d_i on slot $(n - s_i) + 1$ in the starred timetable with C_1. Intuitively speaking, we 'slide' the positions which we colored with C_0 one down, as in Fig. 6. In general, for the color C_k with $1 \leqslant k \leqslant n-1$, we first color the pair of teams assigned to depth 0 in slot k with C_k. Then, for $i = 1, \ldots, m$, we color the two teams of depth d_i on slot $s_i + k$ in the overbar timetable, and the two teams of depth d_i on slot $(n - s_i) + k$ in the starred timetable with C_k. The next claim shows that the procedure described above produces a legitimate coloring.

Claim 10. *In the method described above*, (i) *each team in* $\{\overline{0}, \ldots, \overline{n-1}\} \cup \{0^*, \ldots, (n-1)^*\}$ *receives the color* C_k *exactly once*, (ii) *exactly one pair of teams in each slot is colored by* C_k, *for* k $(1 \leqslant k \leqslant n-1)$, (iii) *each match is assigned exactly one color from* $C_0, C_1, \ldots, C_{n-1}$, *and* (iv) *the repetition matches* $(\overline{0}, 0^*), (\overline{1}, (n-1)^*), \ldots, (\overline{n-1}, 1^*)$ *are given a different color in the first and second halves*.

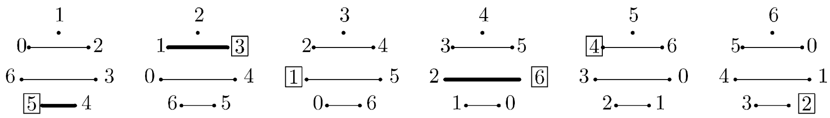

Fig. 5. Choosing lines.

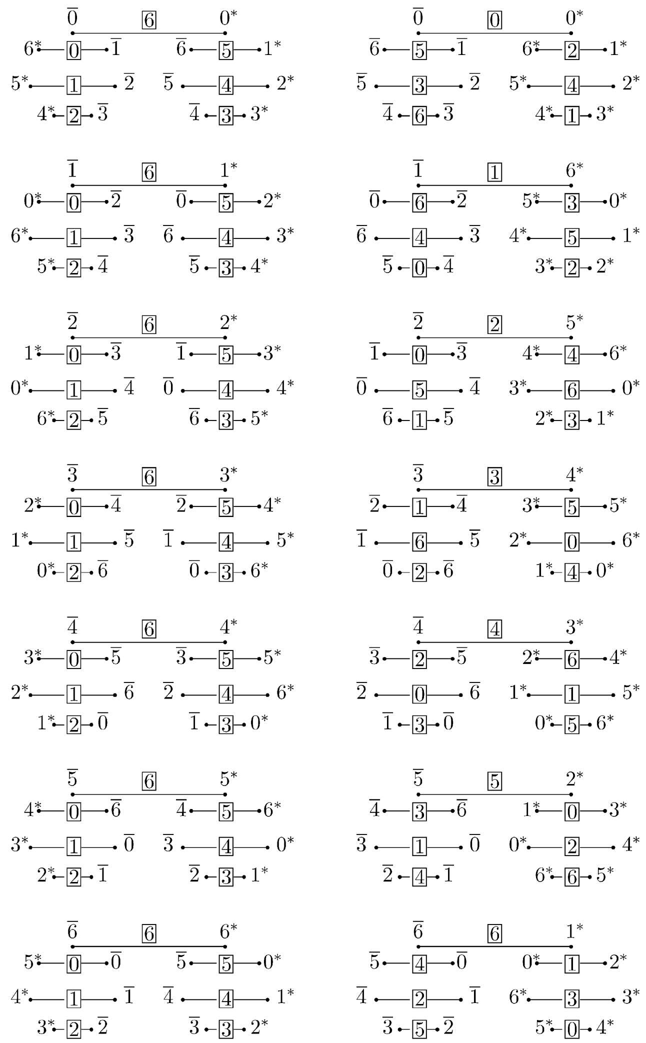

Fig. 6. Solution for 14 teams.

Proof. Let $s_i' \in \{0, 1, \dots, n-1\}$ be the slot with $s_i' \equiv s_i + k \pmod n$. If $\overline{t_{2i-1}'}$ and $\overline{t_{2i}'}$ are the two labels with depth d_i on slot s_i' in the overbar timetable, then by (lab3), t_{2i-1}' and t_{2i}' satisfy

$$t_{2i-1}' \equiv t_{2i-1} + k \pmod n, \qquad t_{2i}' \equiv t_{2i} + k \pmod n.$$

Taking the fact that the label of position 0 on slot s_i' is $\overline{k}$ into account, we see that the set of overbar teams which are colored with C_k is equal to $\{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$. Similarly, the labels of depth d_i (>0) on slot $(n - s_i) + k$ are congruent to $t_{2i-1} - k$ and $t_{2i} - k \pmod n$, and the label of position 0 is $(n-k)^*$, thus it follows that each starred team is also colored by C_k exactly once. The fact that exactly one pair of teams in each slot is colored by C_k is immediate from the choice of $s_1, s_2, \dots, s_m$, and since the line depths $d_1, d_2, \dots, d_m$ are all distinct and greater than 0, each match is assigned exactly one color. Finally, we observe that match $(\overline{i}, (n-i)^*)$ is given color C_{i-1} in the first half and C_i in the second half, completing the proof of the claim.

It now remains to show that we can always choose a set of line-slot pairs with the desired conditions. For this we will use the notion of primitive roots and indices.

Since n is a prime number and $n \geqslant 7$, n has a primitive root r. From Proposition 3, r always satisfies $1 < r < n-1$. Now, let x be the unique number in $\{1, 2, \ldots, n-1\}$ satisfying

$$\mathrm{Ind}_r\, x \equiv \mathrm{Ind}_r(r-1) - \mathrm{Ind}_r(r+1) \pmod{n-1}.$$

Such an x always exists, because $r \neq 1, n-1$. For example, if $n = 7$ and $r = 3$ (see Fig. 3), we must have $\mathrm{Ind}_3\, x \equiv \mathrm{Ind}_3\, 2 - \mathrm{Ind}_3\, 4 = 2 - 4 \equiv 4 \pmod 6$, hence $x = 4$. Obviously, $x \neq 1$. The chosen x also satisfies $x \neq n-1$, because $x = n-1$ implies $\mathrm{Ind}_r(r-1) = \mathrm{Ind}_r(r+1) + \mathrm{Ind}_r(n-1) \pmod{n-1}$, which combined with (ind1) in turn yields $r - 1 \equiv (n-1)(r+1) \equiv -(r+1) \pmod n$, i.e. $2r \equiv 0 \pmod n$.

Now, for $s = 1, 2, \ldots, n-1$, let i_s be the position satisfying $i_s \equiv sx \pmod n$, and for slot s, choose the line incident to the position i_s in the basic timetable for n. In rough terms, this means that we go around the basic polygon in the clockwise direction, choosing every xth label. In Fig. 5, the boxes indicate the labels which are chosen when $n = 7$ and $x = 4$. Note that we never arrive at depth 0, because sx is never a multiple of n, hence we are justified in saying that we choose a line. We will show that we may select $m = (n-1)/2$ line-slot pairs from the $n-1$ pairs described above so the required conditions are satisfied.

Claim 11. *For distinct $s, s' \in \{1, 2, \ldots, n-1\}$, i_s and $i_{s'}$ have the same depth if and only if $s + s' = n$.*

Proof. It is clear that i_s and $i_{s'}$ have the same depth if and only if, $i_s = i_{s'}$ or $i_s + i_{s'} = n$. The former condition is equivalent to $sx \equiv s'x \pmod n$, however, this can never happen because s and s' are distinct, and x is relatively prime to n. On the other hand, the latter condition is equivalent to $(s + s')x \equiv 0 \pmod n$ which in turn is equivalent to $s + s' \equiv 0 \pmod n$. Thus the claim is proved. $\square$

This claim shows that a set of $m = (n-1)/2$ of the above line-slot pairs which satisfy the third condition will always satisfy the second. Thus only the first and last conditions will need further consideration.

We now turn our attention to the first condition, and consider properties of the labels on either side of the chosen lines. By (lab3) we must have $label(i_s, s) \equiv s(x+1) \pmod n$. By (lab4), the label of the 'opposite side' of the chosen line, i.e. $label(n - i_s, s)$ is congruent to $s(n - x + 1) \pmod n$. Because $x + 1$ and n are relatively prime, all labels $label(i_1, 1), \ldots, label(i_{n-1}, n-1)$ are distinct, and different from $0 = 0(x+1)$, the same can be said for the labels of the opposite sides, because $n - x + 1$ and n are relatively prime. We must now consider the case when $label(n - i_s, s) = label(i_{s'}, s')$ or equivalently, $s(n - x + 1) \equiv s'(x+1) \pmod n$ occurs.

Claim 12. *For $s, s' \in \{1, 2, \ldots, n-1\}$, $label(n - i_s, s) = label(i_{s'}, s')$ if and only if $s \equiv rs' \pmod n$.*

Proof. First, note that $s(n - x + 1) \equiv s'(x+1) \pmod n$ is equivalent to

$$\mathrm{Ind}\, s + \mathrm{Ind}(n - x + 1) \equiv \mathrm{Ind}\, s' + \mathrm{Ind}(x+1) \pmod{n-1},$$

by (ind1) and (ind2). By the definition of x,

$$\begin{aligned} &\mathrm{Ind}\, x \equiv \mathrm{Ind}(r-1) - \mathrm{Ind}(r+1) \pmod{n-1} \\ &\quad \Leftrightarrow (r+1)x \equiv r - 1 \pmod n \\ &\quad \Leftrightarrow r(n - x + 1) \equiv x + 1 \pmod n \\ &\quad \Leftrightarrow \mathrm{Ind}\, r + \mathrm{Ind}(n - x + 1) \equiv \mathrm{Ind}(x+1) \pmod{n-1}. \end{aligned}$$

By combining the two results, we obtain

$$\begin{aligned} &\mathrm{Ind}\, r \equiv \mathrm{Ind}\, s - \mathrm{Ind}\, s' \pmod{n-1} \\ &\quad \Leftrightarrow \mathrm{Ind}\, s \equiv \mathrm{Ind}\, rs' \pmod{n-1} \\ &\quad \Leftrightarrow s \equiv rs' \pmod n. \end{aligned}$$

We now select a set of $m = (n-1)/2$ slots in which each $1, 2, \ldots, n-1$ appears exactly once as the label incident to the chosen lines, i.e., a set of line-slot pairs satisfying the first condition. It turns out that this set will satisfy not only the first condition, but also the third, thus completing the proof.

We consider the directed graph that has vertex set $\{1, 2, \ldots, n-1\}$ and an arc from s to s' whenever $label(n - i_s, s) = label(i_{s'}, s')$. (Note that $label(n - i_s, s) = label(i_{s'}, s')$ prohibits both slots s and s' from being present in the choice of the m line-slot pairs.) By Claim 12, this occurs whenever $\operatorname{Ind} s \equiv \operatorname{Ind} s' + 1 \pmod{n-1}$. Since r is a primitive root of p, this implies that the directed graph consists of exactly one directed cycle on $n - 1 = 4q + 2$ vertices, and is hence bipartite. Moreover, since $\operatorname{Ind}(n - s) \equiv \operatorname{Ind} s + (n-1)/2 \pmod{n-1}$ by (ind5), the distance between s and $n - s$ in the graph is equal to $(n-1)/2 = 2q + 1$, an odd number, and s and $n - s$ will always fall in opposite sides of the bipartition. Thus by choosing all the $(n-1)/2$ slots that comprise one side of the bipartition, we have attained our goal. For $n = 7$ and $x = 4$ (the situation shown in Fig. 5), the cycle is $1 \to 5 \to 4 \to 6 \to 2 \to 3 \to 1$, by Fig. 3 and the bipartition can be written as $(\{1, 4, 2\}, \{5, 6, 3\})$. The thick lines shown in Fig. 5 were constructed from the first set.

Remark 13. The method described above will not work for $n = 3$ because the only primitive root of 3 is $2 = n - 1$, meaning that x cannot be defined. It is neither applicable when n is composite, because composite numbers do not have primitive roots possessing the properties described in Section 2.2. In the case of prime numbers which can be written as $4q + 1$, we may compute x and choose lines on slots, however, the distance between s and $n - s$ in the final bipartite graph will be $2q$, forcing s and $n - s$ into the same side of the bipartition. Hence the conditions $n = 4q + 3$, n is prime, and $n \neq 3$ are all essential in our arguments.

Remark 14. As in the case $n = 4q + 1$, our construction can be used to produce a schedule that satisfies condition (sched5′) instead of (sched5). To do this, we change the color assignment in the first half by assigning color C_0 to the two teams of depth 0, and for line $d = 1, 2, \ldots, (n-1)/2$, color C_d in the overbar polygon and color C_{n-d} in the starred polygon. We add that such a schedule cannot be created by applying the results of [8].

Remark 15. In this case, our construction produces a schedule in which all repeated matches take place on a single day, namely, the first. By eliminating this day, we may obtain a balanced tournament design on $2n$ teams.

3.3. The case $n = 4q + 3$ and n is a composite number

In this case, we decompose n into the product of two odd numbers, and create a number of subleagues, each consisting of an odd number of teams. We then use the results of the previous two subsections to first create a 'subleague schedule', then assign individual games between subleagues.

Since $n = 4q + 3$ is a composite number, we can always express n as $n = n_1 n_2$ with $n_1 = 4q_1 + 1$ ($\geqslant 5$) and $n_2 = 4q_2 + 3$ ($\geqslant 3$), because $(4q_1 + 1)(4q_2 + 1) \equiv (4q_1 + 3)(4q_2 + 3) \equiv 1 \pmod 4$. Moreover, n_2 may always be taken as a prime number. Using these two values, we divide the $N = 2n$ teams into $2n_2$ subleagues with n_1 teams each. We will denote these subleagues by $T_0, T_1, \ldots, T_{2n_2-1}$. We also divide the $2n$ slots into $2n_2$ periods of n_1 slots each, and the n colors into n_2 groups of n_1 colors each. The indices of periods and color groups will begin from 0.

We first note that we can create a schedule for $2n_2$ teams on $2n_2$ slots with n_2 colors, that satisfies conditions (sched1) to (sched4) and (sched5′). When $n_2 \geqslant 7$, we use the results of the previous subsection (see Remark 14), and when $n_2 = 3$ an example of such a schedule for the six teams $a, b, \ldots, f$ is provided by the array in Fig. 7. (In the array, rows correspond to colors, columns to slots, and entries show the competing teams.) We regard this schedule on $2n_2$ teams as a 'subleague schedule' and use it to create a timetable and coloring assignment for the original $2n = n_1 \cdot (2n_2)$ teams. To avoid confusion in the following part, we will refer to teams in the above schedule as subleague indices, the slots as

	0	1	2	3	4	5
C_0	$c-e$	$e-f$	$b-f$	$b-c$	$a-d$	$a-d$
C_1	$d-f$	$a-c$	$c-d$	$a-f$	$b-e$	$b-e$
C_2	$a-b$	$b-d$	$a-e$	$d-e$	$c-f$	$c-f$

Fig. 7. Schedule for six teams.

		period 0					period 1				
		0	1	2	3	4	0	1	2	3	4
C_0	0	$0c–0e$	$1c–1e$	$2c–2e$	$3c–3e$	$4c–4e$	$0e–0f$	$1e–1f$	$2e–2f$	$3e–3f$	$4e–4f$
	1	$4c–1e$	$0c–2e$	$1c–3e$	$2c–4e$	$3c–0e$	$4e–1f$	$0e–2f$	$1e–3f$	$2e–4f$	$3e–0f$
	2	$3c–2e$	$4c–3e$	$0c–4e$	$1c–0e$	$2c–1e$	$3e–2f$	$4e–3f$	$0e–4f$	$1e–0f$	$2e–1f$
	3	$2c–3e$	$3c–4e$	$4c–0e$	$0c–1e$	$1c–2e$	$2e–3f$	$3e–4f$	$4e–0f$	$0e–1f$	$1e–2f$
	4	$1c–4e$	$2c–0e$	$3c–1e$	$4c–2e$	$0c–3e$	$1e–4f$	$2e–0f$	$3e–1f$	$4e–2f$	$0e–3f$
C_1	0	$0d–0f$	$1d–1f$	$2d–2f$	$3d–3f$	$4d–4f$	$0a–0c$	$1a–1c$	$2a–2c$	$3a–3c$	$4a–4c$
	1	$4d–1f$	$0d–2f$	$1d–3f$	$2d–4f$	$3d–0f$	$4a–1c$	$0a–2c$	$1a–3c$	$2a–4c$	$3a–0c$
	2	$3d–2f$	$4d–3f$	$0d–4f$	$1d–0f$	$2d–1f$	$3a–2c$	$4a–3c$	$0a–4c$	$1a–0c$	$2a–1c$
	3	$2d–3f$	$3d–4f$	$4d–0f$	$0d–1f$	$1d–2f$	$2a–3c$	$3a–4c$	$4a–0c$	$0a–1c$	$1a–2c$
	4	$1d–4f$	$2d–0f$	$3d–1f$	$4d–2f$	$0d–3f$	$1a–4c$	$2a–0c$	$3a–1c$	$4a–2c$	$0a–3c$
C_2	0	$0a–0b$	$1a–1b$	$2a–2b$	$3a–3b$	$4a–4b$	$0b–0d$	$1b–1d$	$2b–2d$	$3b–3d$	$4b–4d$
	1	$4a–1b$	$0a–2b$	$1a–3b$	$2a–4b$	$3a–0b$	$4b–1d$	$0b–2d$	$1b–3d$	$2b–4d$	$3b–0d$
	2	$3a–2b$	$4a–3b$	$0a–4b$	$1a–0b$	$2a–1b$	$3b–2d$	$4b–3d$	$0b–4d$	$1b–0d$	$2b–1d$
	3	$2a–3b$	$3a–4b$	$4a–0b$	$0a–1b$	$1a–2b$	$2b–3d$	$3b–4d$	$4b–0d$	$0b–1d$	$1b–2d$
	4	$1a–4b$	$2a–0b$	$3a–1b$	$4a–2b$	$0a–3b$	$1b–4d$	$2b–0d$	$3b–1d$	$4b–2d$	$0b–3d$

		period 4					period 5				
		0	1	2	3	4	0	1	2	3	4
C_0	0	$0a–0d$	$1a–1d$	$2a–2d$	$3a–3d$	$4a–4d$	$4d–1a$	$0d–2a$	$1d–3a$	$2d–4a$	$3d–0a$
	1	$4a–1a$	$0a–2a$	$1a–3a$	$2a–4a$	$3a–0a$	$3d–2d$	$4d–3d$	$0d–4d$	$1d–0d$	$2d–1d$
	2	$3d–2a$	$4d–3a$	$0d–4a$	$1d–0a$	$2d–1a$	$4a–1d$	$0a–2d$	$1a–3d$	$2a–4d$	$3a–0d$
	3	$4d–1d$	$0d–2d$	$1d–3d$	$2d–4d$	$3d–0d$	$3a–2a$	$4a–3a$	$0a–4a$	$1a–0a$	$2a–1a$
	4	$3a–2d$	$4a–3d$	$0a–4d$	$1a–0d$	$2a–1d$	$0a–0d$	$1a–1d$	$2a–2d$	$3a–3d$	$4a–4d$
C_1	0	$0b–0e$	$1b–1e$	$2b–2e$	$3b–3e$	$4b–4e$	$4e–1b$	$0e–2b$	$1e–3b$	$2e–4b$	$3e–0b$
	1	$4b–1b$	$0b–2b$	$1b–3b$	$2b–4b$	$3b–0b$	$3e–2e$	$4e–3e$	$0e–4e$	$1e–0e$	$2e–1e$
	2	$3e–2b$	$4e–3b$	$0e–4b$	$1e–0b$	$2e–1b$	$4b–1e$	$0b–2e$	$1b–3e$	$2b–4e$	$3b–0e$
	3	$4e–1e$	$0e–2e$	$1e–3e$	$2e–4e$	$3e–0e$	$3b–2b$	$4b–3b$	$0b–4b$	$1b–0b$	$2b–1b$
	4	$3b–2e$	$4b–3e$	$0b–4e$	$1b–0e$	$2b–1e$	$0b–0e$	$1b–1e$	$2b–2e$	$3b–3e$	$4b–4e$
C_2	0	$0c–0f$	$1c–1f$	$2c–2f$	$3c–3f$	$4c–4f$	$4f–1c$	$0f–2c$	$1f–3c$	$2f–4c$	$3f–0c$
	1	$4c–1c$	$0c–2c$	$1c–3c$	$2c–4c$	$3c–0c$	$3f–2f$	$4f–3f$	$0f–4f$	$1f–0f$	$2f–1f$
	2	$3f–2c$	$4f–3c$	$0f–4c$	$1f–0c$	$2f–1c$	$4c–1f$	$0c–2f$	$1c–3f$	$2c–4f$	$3c–0f$
	3	$4f–1f$	$0f–2f$	$1f–3f$	$2f–4f$	$3f–0f$	$3c–2c$	$4c–3c$	$0c–4c$	$1c–0c$	$2c–1c$
	4	$3c–2f$	$4c–3f$	$0c–4f$	$1c–0f$	$2c–1f$	$0c–0f$	$1c–1f$	$2c–2f$	$3c–3f$	$4c–4f$

Fig. 8. Periods 0, 1, 4 and 5 of solution for 30 teams.

subleague slots, and the colors as subleague colors. Thus, the terms ‘team’, ‘slot’, ‘period’, ‘color’ and ‘color group’ will always refer to those of the original case. Note that subleague slots and subleague colors correspond to period and color group indices, respectively.

Let i and j be two distinct subleague indices. Then, in the subleague schedule, either i and j oppose each other just once on subleague slot s with subleague color k, or they play against each other twice on two distinct subleague slots s and s', both times using the same subleague color k. In the former case, we have the teams in subleagues T_i and T_j play all their interleague games in period s using the colors in color group k. Since n_1 is an odd number, period s consists of n_1 slots, and color group k contains n_1 colors, this can be done by using the method employed to create the first half in Section 3.2. In the latter case we create a schedule satisfying conditions (sched1) to (sched5) for the $2n_1$ teams contained in subleagues T_i and T_j on $2n_1$ slots, using the n_1 colors in color group k. This can be accomplished by using the method described in Section 3.1. We then allot the $2n_1$ slots equally among periods s and s'. Note that the repetition matches will always arise in the latter case.

Fig. 8 shows periods 0, 1, 4 and 5 of the resulting schedule for the case $2n = 30$ ($n = 15$). Since $n_1 = 5$ and $n_2 = 3$, there are six subleagues of five teams each, six periods of five slots each, and three color groups of five colors each. In the figure, the six subleagues are called $a, b, \ldots, f$, and teams in each subleague are specified by a number-alphabet pair; subleague a consists of the five teams $\{0a, 1a, \ldots, 4a\}$, and so on. As in Fig. 7, rows correspond to colors and columns to slots. In the schedule for six teams given in Fig. 7, we see that the match held in slot 0 with color C_0 is $c - e$, a nonrepeat game. Thus, in the schedule for 30 teams, subleagues c and e hold all their interleague games in period 0, using the five colors in color group C_0 (see Fig. 2). The correspondences between other games in periods 0 and 1 and the schedule for 6 teams should be obvious. On the other hand, the match $a - d$ in Fig. 7 is a repeat game played on slots 4 and 5 with color C_0. Accordingly, the schedule for the teams in subleagues a and d in periods 4 and 5 is identical to that given in Fig. 4 with the overbar and starred teams replaced by the teams in subleagues a and d, and the colors used understood to belong to color group C_0.

In the completed schedule for $2n$ teams, it is easily seen that each team pair plays against each other at least once. Also, since each subleague index is assigned each subleague color exactly twice, each team uses each color exactly twice. Finally, since the repetition games were assigned by the procedure given in Section 3.1, we see that no pair of teams is assigned the same color twice. Thus, we have successfully created a schedule for $2n$ teams when $n = 4q + 3$ is a composite number.

Summing up the results of this section, we have proved the following theorem stated in Section 1.

Theorem 1. *Let n be any odd number greater than or equal to* 5. *Then there always exists a schedule for* 2*n teams and n stadiums satisfying conditions* (*sched1*) *to* (*sched5*).

Remark 16. As in the previous two subcases, we may substitute the condition (sched5$'$) for (sched5) when $n = 4q + 3$ is a composite number, because the repeated games were constructed by the method in Section 3.1 and Remark 7.

Taking the cases $n = 3$ and 1 into account, we have also proved the following statement.

Corollary 17. *Let n be any* (*positive*) *odd number*. *Then there always exists a schedule for* 2*n teams and n stadiums satisfying conditions* (*sched*1) *to* (*sched*4) *and* (sched5$'$).

4. The case N=2n with n even

We now turn to the case when n is even. Since the case where $2n$ is equal to a power of two has been proved in [3], we may concentrate on cases when this does not hold, i.e., $2n$ may be expressed as $2n = 2^q m$, $q \geqslant 2$ and m is an odd number greater than or equal to 3. We divide this into two cases; the first when $q \geqslant 3$ and the second when $q = 2$.

4.1. The case $2n = 2^q m$, $q \geqslant 3$ and $m \geqslant 3$ is odd

In this case we combine the results of [3] with the method used in Section 3.3. First, we divide the $2n$ teams into $2m$ subleagues of 2^{q-1} teams each, then create $2m$ periods and m color groups, each having 2^{q-1} slots and 2^{q-1} colors, respectively.

Since m is odd, we may construct a ‘subleague schedule’ on the $2m$ subleague indices satisfying conditions (sched1) to (sched4), and (sched 5$'$), by Corollary 17. Interleague schedules for two subleagues can be constructed by Proposition 3 in [3] because all subleagues contain $2^{q-1} \geqslant 4$ teams. Also, since the total number of teams in two subleagues is equal to $2^q \geqslant 8$, we can again apply the results of de Werra et al. [3] to obtain appropriate schedules for the subleague indices which are repetitions. This completes the case when $q \geqslant 3$.

Unfortunately, when $q = 2$, each subleague will have two teams, and the above method will not work. Thus, we treat this case separately in the following subsection.

4.2. The case $2n = 4m$ and $m \geqslant 3$ is odd

As we have just mentioned above, this case is somewhat of an anomaly that must be given special treatment. We have tried to give a method as simple and systematical as possible, however, there are seem to be unique difficulties

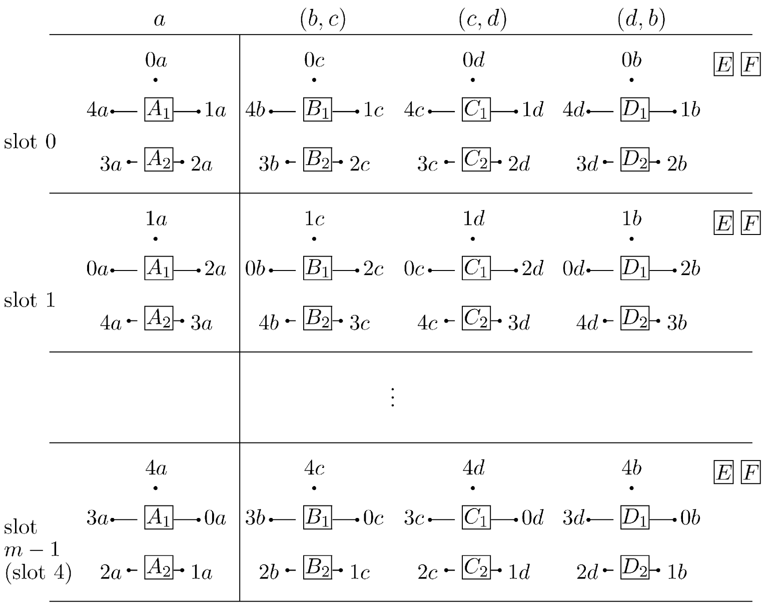

Fig. 9. Basic idea for period 1.

which arise from the fact that a schedule for four teams (with or without condition (sched5)) does not exist, and the final procedure is somewhat complicated. We will rely a great deal on the figures in describing the details.

We divide the $4m$ teams into four subleagues of m teams each, and call them a, b, c, and d. Teams in each subleague will be specified by a number-alphabet pair. For example, the set of teams in subleague a is $\{0a, 1a, \ldots, (m-1)a\}$. We also separate the $4m$ slots into four periods of m slots each, and the $2m$ colors into four groups of $(m-1)/2$ colors each, plus two additional colors. The periods will be numbered from 1 to 4, and slots in each period from 0 to $m-1$. Individual slots will be identified by the pair of period and slot numbers, for example, (1, 0) will mean slot 0 of period 1. We call the four color groups A, B, C, and D, and give the elements of each color group indices from 1 to $k = (m-1)/2$; for example $A = \{A_1, A_2, \ldots, A_k\}$. We will call the two additional colors E and F.

The basic idea of our construction is shown in Fig. 9 (where $m = 5$). In each period, we have one of the four subleagues playing all its internal games, while the remaining subleagues play interleague games. Consider period 1, and let a be the subleague playing internal games. We create four basic timetables for m, and designate one of them to be used for the games internal to a. In Fig. 9, the four timetables are shown by four columns which are numbered from the left as 1, 2, 3 and 4. The timetable we allot to a is the leftmost one, column 1. In this timetable, we assign labels by the following method: if $label(i, s) = t$ in the basic timetable, then we replace t with ta for all positions $i = 0, 1, \ldots, m-1$ and all slots $s = 0, 1, \ldots, m-1$.

Next, we choose three *ordered pairs* of the remaining subleagues b, c, and d, in which each subleague appears once as the first element, and once as the second element. In Fig. 9, the chosen pairs are (b, c), (c, d) and (d, b). We then assign each of the ordered pairs to the remaining three basic timetables (in Fig. 9, pair (b, c) is assigned to column 2, pair (c, d) to column 3, and (d, b) to column 4), and give labels to positions in each polygon. In each of the three timetables, positions to the left of each line are given labels belonging to the first element of the assigned ordered pair, and the remaining positions are given labels from the second element. For example, the second column in Fig. 9 was assigned the ordered pair (b, c), thus positions 3 and 4 have labels with 'b', and positions 0, 1, and 2 have labels with 'c'. Note that this second timetable corresponds to 'one side' of the complete interleague timetable for subleagues b and c, constructed by the method used for the first half in Section 3.2.

Forgetting, for the moment, about opponents of teams assigned to position 0, we now give a color assignment for period 1. For $d = 1, 2, \ldots, k = (m-1)/2$, assign color A_d to line d in column 1, color B_d to line d in column 2, and so on. The colors E and F will be reserved for depth 0. Then, the following claim on the timetable and coloring assignment for period 1 holds:

		column		
period	1	2	3	4
1	a	(b, c)	(c, d)	(d, b)
2	b	(d, c)	(a, d)	(c, a)
3	c	(d, a)	(a, b)	(b, d)
4	d	(b, a)	(c, b)	(a, c)

Fig. 10. Assignment for subleague pairs and columns.

Claim 18. *In the first period, the following facts hold*: (i) *all games internal to subleague a have been held*, (ii) *each team of subleague a uses each color in group A twice, and* (iii) *each team in subleague b uses the colors in groups B and D exactly once, each team in subleague c uses the colors in groups B and C exactly once, likewise each team in subleague d uses the colors in groups C and D exactly once.*

Continuing to defer the opponents and colors of team labels with depth 0 till afterward, we now consider timetables and color assignments for the remaining second, third, and fourth periods (see Fig. 10). We begin by setting the subleague playing internal games to b in period 2, c in period 3, and d in period 4. Next is the designation of ordered pairs to columns. In order to ensure that all interleague games between, for example, b and c are indeed held, we need to assign the ordered pair (c, b) that was not chosen in period 1, to a column in period 4, likewise, (d, c) should appear in period 2, and (b, d) in period 3. In fact, the essential conditions may be stated as:

- in all periods, each team that is not assigned to the first column appears once as the first element, and once as the second element in the ordered pairs assigned to columns 2, 3, and 4,
- both orderings of each subleague pair are assigned somewhere in the four periods,
- all subleagues are assigned twice to each column 2, 3, and 4.

The assignment given in Fig. 10 is one such example. Finally, we consider the color assignment. If we use the same scheme of color-line association described for period 1 throughout the remaining periods, the final condition stated above guarantees that all teams use each color in groups A, B, C, and D twice. The timetable and color assignment constructed so far, satisfy the following claim:

Claim 19. *The following facts hold*: (i) *all games internal to each subleague have been scheduled*, (ii) *all interleague games have been scheduled with the exception of games between two teams with the same number, i.e., games between teams in* $\{0a, 0b, 0c, 0d\}$, *teams in* $\{1a, 1b, 1c, 1d\}$ *and so on*, (iii) *all teams use each color in groups A, B, C, and D twice.*

We now turn to the task of assigning opponents and colors to the teams at depth 0. Note that in all periods, these will be $\{0a, 0b, 0c, 0d\}$ on slot 0, $\{1a, 1b, 1c, 1d\}$ on slot 1, and so on. If we could create a schedule for four teams which satisfies conditions (sched1) to (sched5), then we would be done, unfortunately this is impossible. Instead, what we do is generate a timetable and color assignment that satisfies the requirements as nearly as possible for the four teams $\{sa, sb, sc, sd\}$ on the four slots $(1, s)$, $(2, s)$, $(3, s)$ and $(4, s)$ $(s = 0, 1, \ldots, m - 1)$ using colors E and F, then make adjustments. In the following explanation we will fix $s = 0$, but the construction is the same for all s.

We temporarily create the timetable and color assignment for $\{0a, 0b, 0c, 0d\}$ on slots $(1, 0), \ldots, (4, 0)$ given in Fig. 11 (labels and colors on lines of depth 1 are also shown). This schedule satisfies all conditions from (sched1) to (sched5) except for (sched4); team $0a$ uses color E three times, but color F only once, conversely teams $0b$ uses color

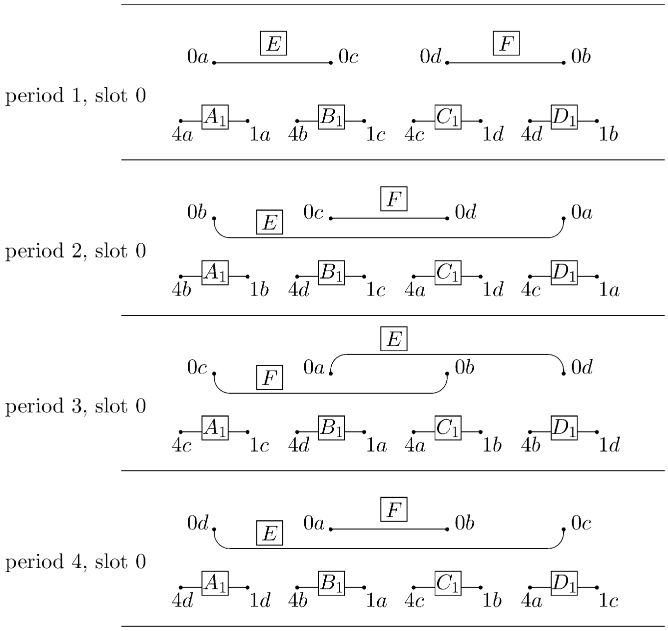

Fig. 11. Temporary schedule for slot 0 of periods 1, 2, 3 and 4 ($m = 5$).

	depth 0		line 1	
1	$(a, c) : E$	$(b, d) : F$	$(c, d) : C_1$	$(d, b) : D_1$
2	$(a, b) : E$	$(c, d) : F \to C_1$	$(a, d) : C_1 \to F$	$(c, a) : D_1$
3	$(a, d) : E \to D_1$	$(b, c) : F$	$(a, b) : C_1$	$(b, d) : D_1 \to E$
4	$(c, d) : E$	$(a, b) : F \to C_1$	$(c, b) : C_1 \to D_1$	$(a, c) : D_1 \to F$

Fig. 12. Color switches.

F three times but color E only once (teams $0c$ and $0d$ both use colors E and F twice each). To remedy this imbalance, we perform the following switches involving colors C_1 and D_1 in all slots for the specified periods (see Fig. 12):

- in period 2, switch colors F and C_1,
- in period 3, switch colors E and D_1,
- in period 4, change color F to C_1, color C_1 to D_1 and color D_1 to F.

Claim 20. *After the above switch is executed, all teams use each color exactly twice.*

Proof. It is sufficient to check only the colors that appear in the switch, namely C_1, D_1, E and F. We first see what happens to the colors used by $0a$, $0b$, $0c$, and $0d$. We observe that team $0a$:

- loses C_1 and gains F on slot (2, 1),
- loses E and gains D_1 on slot (3, 0),

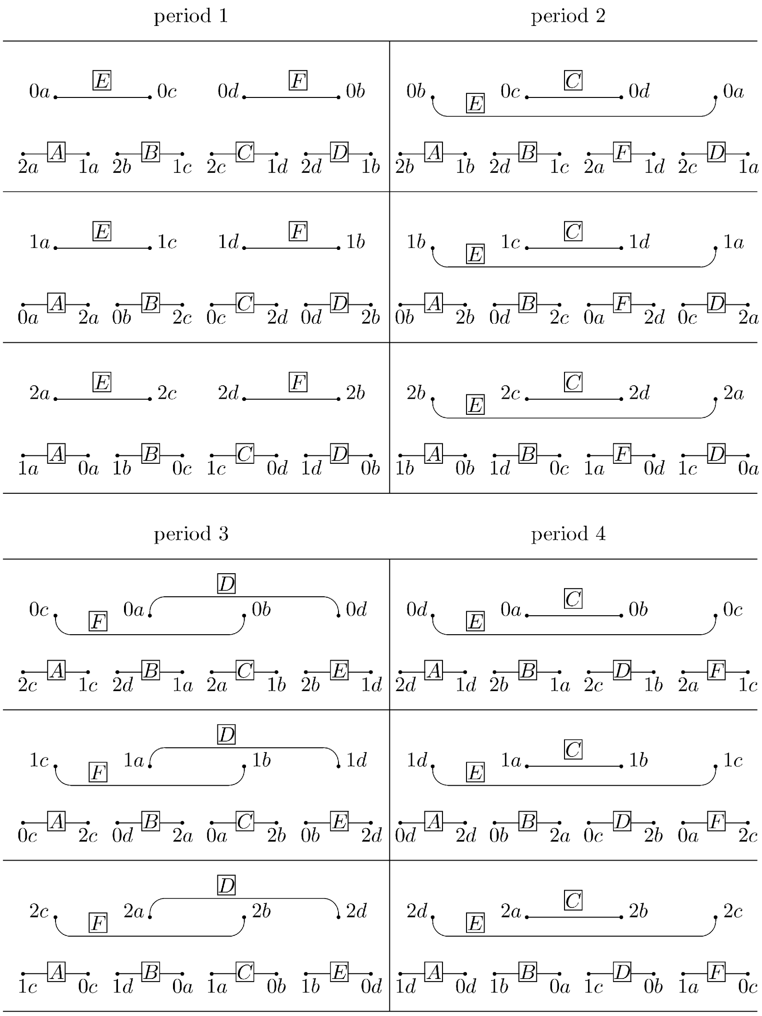

Fig. 13. Schedule for 12 teams.

- loses F and gains C_1 on slot (4, 0),
- loses D_1 and gains F on slot (4, 1).

Thus, in total, $0a$ loses one instance of E and gains one instance of F. Similarly, $0b$:

- loses D_1 and gains E on slot (3, 1),
- loses F and gains C_1 on slot (4, 0),
- loses C_1 and gains D_1 on slot $(4, m - 1)$,

and in total, loses one instance of F and gains one instance of E. Now, $0c$:

- loses F and gains C_1 on slot (2, 0),
- loses C_1 and gains D_1 on slot (4, 1),
- loses D_1 and gains F on slot $(4, m - 1)$,

and $0d$:

- loses F and gains C_1 on slot (2, 0),
- loses C_1 and gains F on slot $(2, m - 1)$,

- loses E and gains D_1 on slot $(3, 0)$,
- loses D_1 and gains E on slot $(3, m - 1)$.

Thus there are no changes in the total usage of colors for either $0c$ or $0d$. Similar statements hold for all other teams, and the claim follows. □

For the sake of completeness, we give the schedule for 12 teams ($m = 3$) in its entirety in Fig. 13. Since all color groups consist of a single color, we omit indices.

Summarizing the results for even n, we have proved the following theorem stated in Section 1:

Theorem 2. *Let n be any even number which is not a power of two. Then there always exists a schedule for* $2n$ *teams and n stadiums satisfying conditions* (*sched*1) *to* (*sched*5).

5. Concluding remarks

We have proved the existence of schedules satisfying conditions (sched1) to (sched5), for values of n not treated in [3], thus providing a complete theoretical solution. The construction is based on six cases: the case when n is a power of two, treated in [3], the case when $n = 4q + 1$, the case when $n = 4q + 3$ and n is prime, the case when $n = 4q + 3$ and n is composite, the case when n is an odd number times a power of two greater than or equal to four, and finally the case when n is twice an odd number. All of these cases use different tools and techniques. It would be nice if the construction could be simplified.

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